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On light propagation in premetric electrodynamics: the covariant dispersion relation

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Abstract

The premetric approach to electrodynamics provides a unified description of a wide class of electromagnetic phenomena. In particular, it involves axion, dilaton and skewon modifications of the classical electrodynamics. This formalism also emerges when the non-minimal coupling between the electromagnetic tensor and the torsion of Einstein–Cartan gravity is considered. Moreover, the premetric formalism can serve as a general covariant background of the electromagnetic properties of anisotropic media. In the current paper, we study wave propagation in the premetric electrodynamics. We derive a system of characteristic equations corresponded to premetric generalization of the Maxwell equation. This singular system is characterized by the adjoint matrix which turns to be of a very special form—proportional to a scalar quartic factor. We prove that a necessary condition for the existence of a non-trivial solution of the characteristic system is expressed by a unique scalar dispersion relation. In the tangential (momentum) space, it determines a fourth-order light hypersurface which replaces the ordinary light cone of the standard Maxwell theory. We derive an explicit form of the covariant dispersion relation and establish its algebraic and physical origin.

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1. Introduction

On classical and quantum levels, Maxwell's electrodynamics is a well-established theory, whose results are in a very precise coordination with the experiment. This theory, however, can require some principal modifications in order to include non-trivial interactions with other physical fields. The following non-complete list indicates some directions of the possible alternations.

- *Dilaton field.* This scalar partner of the classical electromagnetic field is proposed recently as a source of a possible variation of the fine-structure constant [1, 2].
- *Axion field.* This pseudo-scalar field is believed to play a central role in violation of Lorentz and parity invariance [3–8].
- *Birefringence and optical activity of vacuum.* These effects are forbidden in the standard (minimal coupling) model of interaction between the electromagnetic and the gravitational fields. In the Cartan–Einstein model of gravity, the non-minimal coupling yields, in general, the non-trivial effects of electromagnetic wave propagation [9–12].

Although the mentioned problems belong to rather different branches of the classical field theory, their joint treatment can be provided in a unique framework of the *premetric electrodynamics*. The roots of such an approach can be found in the older literature [13], but its final form was derived only recently, see [14–20] and specially the book [19] and the references given therein.

In the premetric construction, the electromagnetic field is considered on a bare differential manifold without metric and/or connection. Instead, the manifold is assumed to be endowed with a fourth-order constitutive pseudo-tensor, which provides the constitutive relation between the electromagnetic field strength and the electromagnetic excitation tensors. The metric tensor itself is only a secondary quantity in this construction. Its explicit form and even its signature is derived from the properties of the constitutive tensor [15–21].

In the current paper, we study the electromagnetic waves propagation in the premetric electrodynamics. From the technical point of view, our approach is similar to those used in the relativistic plasmodynamics [24–26]. A principal difference is that we are dealing with a metric-free background, thus the norm of the wave covector and its scalar product with another covector are not acceptable. Roughly speaking, the indices in the tensorial expressions cannot be raised or lowered. Moreover, we show that, for the electromagnetic waves propagation, the metric tensor is indeed a secondary structure. The metric structure must be considered as a result of the properties of the wave propagation and not as a predeclared fact.

The main result of our consideration is a rigorous derivation of the covariant dispersion relation. It is shown to originate from the adjoint matrix of a characteristic system of the generalized Maxwell field equations.

The organization of the paper is as follows. In section 1, we give a brief account of the premetric electrodynamics formalism. Section 2 is devoted to the geometric optics approximation and the wave-type ansatz. When this ansatz is substituted in the Maxwell system, the former is transformed into a system of linear algebraic equations. The algebraic features of this characteristic system are studied in section 4. A covariant dispersion relation emerges in section 5 as a necessary condition for the existence of a non-trivial wave-type solution in premetric Maxwell electrodynamics. In section 6, we give an application of the developed formalism to the simplest Maxwell case. The standard expressions of the Maxwell theory are reinstated. Section 7 is devoted to a discussion of the proposed formalism and its possible generalizations.

2. Premetric electrodynamics formalism

2.1. Motivations

In order to represent the motivations of the premetric electrodynamics, we briefly recall some electromagnetism models which naturally lead to this generalization.

2.1.1. Vacuum electrodynamics. In flat Minkowski spacetime, the electromagnetic field is described by the antisymmetric tensor of the electromagnetic field strength F_{ij}. In an orthogonal Cartesian coordinate system $\{x^i\}$ with $i = 0, 1, 2, 3$, the dynamics of the field is defined from a pair of the first-order partial differential equations:

$$\epsilon^{ijkl} F_{jk,l} = 0, \qquad F^{ij}{}_{,j} = J^i. \tag{2.1}$$

Here, the comma denotes the partial differentiation. The Lévi-Civitá permutation pseudo-tensor ϵ^{ijkl} with the values $\{-1, 0, 1\}$ is normalized by $\epsilon^{0123} = 1$.

The first equation of (2.1) is completely independent of the metric. In the second one, the Minkowski metric, $\eta^{ij} = \mathrm{diag}(-1, 1, 1, 1)$, is involved implicitly. It is used here for the definition of the covariant components of the field strength, i.e for raising the indices

$$F^{ij} = \eta^{im}\eta^{jn} F_{mn}. \tag{2.2}$$

To have a representation similar to those used below, we rewrite this equation as

$$F^{ij} = \tfrac{1}{2}\chi^{ijmn} F_{mn}, \qquad \text{where} \quad \chi^{ijmn} = \eta^{im}\eta^{jn} - \eta^{in}\eta^{jm}. \tag{2.3}$$

In (2.1), the vector field J^i describes the electric current. Since the tensor F^{ij} is antisymmetric, the electric charge conservation law

$$J^i{}_{,i} = 0 \tag{2.4}$$

is a straightforward consequence of (2.1).

The relations above are invariant under a subgroup of linear rigid transformations of coordinates which preserve the specific form of the Minkowski metric. This group includes the instantaneous spatial rotations, Lorentz's transformations and reflections.

2.1.2. Electrodynamics in gravity field. In a non-inertial frame, i.e. in curvilinear coordinates on the flat spacetime, the Minkowski metric η^{ij} is replaced by a generic pseudo-Riemannian metric g^{ij} whose components depend on a spacetime point. On this background, the transformational requirements are changed. The field equations must now be invariant under arbitrary smooth transformations of the coordinates. To satisfy this transformational requirement, the field equations (2.1) are modified to

$$\epsilon^{ijkl} F_{jk,l} = 0, \qquad (F^{ij}\sqrt{-g})_{,j} = J^i\sqrt{-g}, \tag{2.5}$$

where $g = \det(g_{ij})$. The covariant components of the electromagnetic field strength are now defined via a multiplication by the metric tensor components which, in contrast to (2.2), depend on a point

$$F^{ij} = g^{im} g^{jn} F_{mn}. \tag{2.6}$$

Observe that now the components of two electromagnetic fields F^{ij} and F_{mn} are different functions of a point. In fact they can be treated as two independent physical fields. In such an approach, metric tensor comes from a relation between these independent fields, i.e. from a physical phenomenon.

Since F^{ij} is antisymmetric, the inhomogeneous field equation of (2.5) yields a modified electric charge conservation law

$$(J^i\sqrt{-g})_{,i} = 0. \tag{2.7}$$

In fact, this equation is not a conservation law for the vector field J^i itself. What is really conserved is the product of J^i with the root of the metric determinant. It means that a conserved electric current cannot be described by a covariant vector field so a redefinition of this basic notion is necessary. Instead of treating it as a vector field, the electric current has to be considered as a weight (+1) pseudo-vector density field

$$\mathcal{J}^i = J^i\sqrt{-g}. \tag{2.8}$$

Also a weight (+1) pseudo-tensor density of electromagnetic excitation

$$\mathcal{H}^{ij} = F^{ij}\sqrt{-g} \tag{2.9}$$

has to be involved. Under smooth transformations of the coordinates $x^i \to x^{i'}$ with the Jacobian $L = \det(\partial x^{i'}/\partial x^i)$, the transformation law for these pseudo-tensorial quantities involves an additional factor $1/|L|$. In order to have a covariant field equation, this factor must be compensated. The first-order partial derivatives of the term $\sqrt{-g}$ make the job and the whole equation is covariant.

Consequently, the general covariant field equations take the form

$$\epsilon^{ijkl}F_{jk,l} = 0, \qquad \mathcal{H}^{ij}{}_{,j} = \mathcal{J}^i, \tag{2.10}$$

while the general covariant charge conservation law is written as

$$\mathcal{J}^i{}_{,i} = 0. \tag{2.11}$$

The constitutive relation between two basic fields takes the form

$$F^{ij} = \tfrac{1}{2}\chi^{ijmn}F_{mn} \tag{2.12}$$

where the constitutive pseudo-tensor

$$\chi^{ijmn} = \sqrt{-g}(g^{im}g^{jn} - g^{in}g^{jm}) \tag{2.13}$$

is involved.

Although the described modification serves the curvilinear coordinates on a flat manifold, it is well known to be enough also for the description of the electromagnetic field in a curved spacetime of GR. Both field equations (2.10) and the conservation law (2.11) are general covariant even being written via the ordinary partial derivatives.

2.1.3. Electrodynamics in anisotropic media. For anisotropic media in a flat Minkowski space, Maxwell's electrodynamics is described by two pairs of 3D vectors E_α, B^α and D^α, H_α, where the Greek indices are assumed to obtain the spatial values, $\alpha, \beta, \ldots = 1, 2, 3$. In the 4D notation, these vectors are assembled into two antisymmetric tensors: the electromagnetic strength tensor F_{ij} with the components

$$F_{0\alpha} = E_\alpha, \qquad F_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma}B^\gamma, \tag{2.14}$$

and the electromagnetic excitation tensor H^{ij} with the components

$$H^{0\alpha} = D^\alpha, \qquad H^{\alpha\beta} = \epsilon^{\alpha\beta\gamma}H_\gamma. \tag{2.15}$$

In the 4D notation, the Maxwell field equations for the electromagnetic field in anisotropic media are written in the form

$$\epsilon^{ijkl}F_{ij,k} = 0, \qquad H^{ij}{}_{,j} = J^i. \tag{2.16}$$

An additional ingredient, the constitutive relation between two electromagnetic tensors, F_{ij} and H^{ij}, describes the characteristic properties of the media. For a wide range of anisotropic materials, a linear constitutive relation is a sufficiently good approximation:

$$D^\alpha = \varepsilon^{\alpha\beta}E_\beta + \gamma^\alpha{}_\beta B^\beta, \qquad H_\alpha = \mu^{-1}_{\alpha\beta}B^\beta + \tilde{\gamma}_\alpha{}^\beta E_\beta. \tag{2.17}$$

The electromagnetic current conservation law $J^i{}_{,i} = 0$ is a consequence of the field equation (2.16). Note also that equations (2.16) are invariant under arbitrary constant linear transformations of the coordinates.

2.2. Premetric field equations

The models accounted above show some similarity.

- The electromagnetic field is described by two second-order antisymmetric tensors. In differential form notation, the field is represented by two second-order differential forms—one twisted and one untwisted.
- All the models are described by similar systems of two first-order partial differential equations.
- The antisymmetric tensorial fields are related by a linear constitutive relation.
- Even in vacuum electrodynamics, the metric of the manifold emerges only via a special four-component tensor, i.e. it plays only a secondary role.

The accounted similarity naturally leads to a premetric generalization of the classical electrodynamics. For a comprehensive account of this subject, see [19, 20] and the references given therein.

In the premetric approach, two differential field equations for two second-order differential forms, the *electromagnetic field strength* F and the *electromagnetic excitation* $\mathcal{H}$, are postulated:

$$\mathrm{d}F = 0, \qquad \mathrm{d}\mathcal{H} = \mathcal{J}. \tag{2.18}$$

In (2.18), F is an even (untwisted) differential form. It does not change under arbitrary transformations of coordinates. Alternatively, $\mathcal{H}$ and $\mathcal{J}$ are odd (twisted) differential forms. Under a change of coordinates with a Jacobian $L = \det\big(L^i{}_j\big)$, they are multiplied by the sign factor of L. Namely such identification of the electromagnetic fields guarantees the proper integral conservation laws for magnetic flux and electric current, see [19].

Both equations (2.18) are expressed via differential forms thus they are manifestly invariant under arbitrary smooth coordinate transformations. In a more general setting, see [19], these equations can be considered as consequences of two integral conservation laws: one for the magnetic flux and one for the electric current.

In a coordinate chart, we represent the forms as

$$F = \tfrac{1}{2} F_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j, \qquad \mathcal{H} = \tfrac{1}{2} \mathcal{H}^{ij} \epsilon_{ijmn}\,\mathrm{d}x^m \wedge \mathrm{d}x^n, \tag{2.19}$$

while

$$\mathcal{J} = \frac{1}{3!} \mathcal{J}^i \epsilon_{ijmn}\,\mathrm{d}x^j \wedge \mathrm{d}x^m \wedge \mathrm{d}x^n. \tag{2.20}$$

Thus, the components F_{ij} constitute an ordinary antisymmetric tensor while the components $\mathcal{H}^{ij}$ and $\mathcal{J}^i$ are pseudo-tensor densities of weight (+1).

Applying the exterior derivatives to (2.19) we rewrite the field equations (2.18) in the tensorial form

$$\epsilon^{ijkl} F_{jk,l} = 0, \qquad \mathcal{H}^{ij}{}_{,j} = \mathcal{J}^i. \tag{2.21}$$

Even being written via the ordinary partial derivatives, these equations are covariant under arbitrary smooth transformations of coordinates. Note also that a metric tensor or a connection is not involved in the construction above. One can even say that these structures are not defined (yet) on the space. In this sense, the construction is premetric. Particularly, in such an approach, the covariant components of the field strength tensor and the contravariant components of the excitation tensor cannot be introduced—the indices cannot be raised or lowered.

2.3. Constitutive relation

The system (2.21) involves 8 equations for 12 independent variables so it is undetermined. Moreover, the 2-form H describes a field generated by a charged source, while the 2-form F describes some other field which acts on a test charge. In a current stage of the construction, these two fields are formal and completely independent. It means that an interaction between two charges is not yet involved. In order to close the system and to involve an interaction, a constitutive relation between the fields F_{ij} and $\mathcal{H}^{ij}$ must be implicated. The simplest choice of a local linear homogeneous relation

$$\mathcal{H}^{ij} = \tfrac{1}{2}\chi^{ijkl}F_{kl} \tag{2.22}$$

is wide enough to describe the most observation data of the ordinary electrodynamics and even involves some additional electromagnetic effects, such as axion, dilaton and skewon partners of photon, see [19]. Also for the electromagnetism into the non-magnetized media, the linear constitutive relation is a good approximation. For the nonlinear extensions of the premetric approach, see [23]. The non-local constitutive relations were considered recently in [22].

Recall that the physical space is considered as a bare manifold without metric or connection. All the information on its geometry is encoded into the constitutive pseudo-tensor χ^{ijkl} which can depend on the time and position coordinates. By definition, this pseudo-tensor inherits the symmetries of the antisymmetric tensors F_{ij}, $\mathcal{H}^{ij}$. In particular,

$$\chi^{ijkl} = \chi^{[ij]kl} = \chi^{ij[kl]}. \tag{2.23}$$

Thus, in general, the fourth-order constitutive tensor χ^{ijkl} has 36 independent components instead of 4^3. Due to the Young diagrams analysis, under the group of linear transformations, such a tensor is irreducibly decomposed into a sum of three independent pieces:

$$\chi^{ijkl} = {}^{(1)}\chi^{ijkl} + {}^{(2)}\chi^{ijkl} + {}^{(3)}\chi^{ijkl}. \tag{2.24}$$

The *axion part* (1 component) and the *skewon part* (15 components) are defined respectively as

$${}^{(3)}\chi^{ijkl} = \chi^{[ijkl]}, \qquad {}^{(2)}\chi^{ijkl} = \tfrac{1}{2}(\chi^{ijkl} - \chi^{klij}). \tag{2.25}$$

The remainder ${}^{(1)}\chi^{ijkl}$ is a *principal part* of 20 independent components. One can also extract a scalar factor from the principal part which was identified recently [22] with the dilaton partner of electromagnetic field [1, 2].

3. Approximation and ansatz

3.1. Semi-covariant approximation

When the constitutive relation (2.22) is substituted into the field equation (2.21_b), we remain with

$$\tfrac{1}{2}\chi^{ijkl}F_{kl,j} + \tfrac{1}{2}\chi^{ijkl}{}_{,j}F_{kl} = \mathcal{J}^i. \tag{3.1}$$

The first term here describes how the electromagnetic field changes in a spacetime of constant media characteristics. Alternatively, the second term describes the spacetime variation of the media characteristics for a constant electromagnetic field. In this paper, we restrict to the *geometrical optics approximation*. In particular, we neglect with the second term of (3.1) relative to the first one. In other words, we restrict to media whose characteristics change slowly on the characteristic distances of the change of the fields. Note that such approximation is not always applicable. In particular, the Carroll–Field–Jackiw modification of electrodynamics [4] can be reformulated as a premetric electrodynamics [6–8] where the first term of (3.1) vanishes, whereas the birefringence effect comes from the second term.

In the framework of the geometrical approximation, we remain with a system of eight equations for six independent components of the electromagnetic field strength F_{ij}:

$$\epsilon^{ijkl}F_{jk,l}=0, \qquad \tfrac{1}{2}\chi^{ijkl}F_{kl,j}=\mathcal{J}^i. \tag{3.2}$$

In a special case of the Maxwell constitutive tensor, the approximation used here yields the inhomogeneous field equation of the form

$$g^{ik}g^{jl}F_{kl,j}=\mathcal{J}^i. \tag{3.3}$$

This is an approximation of the covariant equation (2.5) when the derivatives of the metric tensor are considered to be small relative to the derivatives of the electromagnetic field. Such a *semi-covariant approximation* is preserved for arbitrary coordinate transformations with small spacetime derivatives.

3.2. Eikonal ansatz

To describe the wave-type solutions of the field equations (3.2), we consider an *eikonal ansatz*. Let the electric current be given in the form

$$J^i=j^ie^\sigma, \tag{3.4}$$

and let the corresponding field strength be expressed as

$$F_{ij}=f_{ij}e^\sigma. \tag{3.5}$$

Here the eikonal σ is a scalar function of a spacetime point. The tensors j^i and f_{ij} are assumed to be slow functions of a point. The derivatives of σ give the main contributions to the field equations. Define the *wave covector*

$$q_i=\frac{\partial\sigma}{\partial x^i}. \tag{3.6}$$

In this approximation, the conservation law for the electric current $J^i{}_{,i}=0$ takes a form of an algebraic equation

$$j^iq_i=0. \tag{3.7}$$

Substituting (3.4), (3.5) into the field equations (3.2) and removing the derivatives of the amplitudes relative to the derivative of the eikonal function, we come to an algebraic system

$$\epsilon^{ijkl}q_jf_{kl}=0, \qquad \chi^{ijkl}q_jf_{kl}=2j^i. \tag{3.8}$$

The same system was derived in [19] by mean of Hadamar's discontinuity propagation method. This fact indicates that a simple approximation used here is not less general than the one used in [19].

Observe a remarkable property of (3.8). If all the quantities involved here are assumed to transform by ordinary tensorial transformation rules with arbitrary pointwise matrices, both equations are preserved. In other words, these equations are straightforward expanded to a general covariant system. This is in spite of the fact that the approximations used in their derivation are not covariant. This property is generic for quasi-linear systems whose leading terms (the higher order derivatives expressions) are linear and thus preserve their form even under arbitrary pointwise transformations.

4. Characteristic equations

4.1. The linear system

The approximation (3.2) and the wave-type ansatz (3.4), (3.5) yield a linear system (3.8) of eight equations for six independent variables. This algebraic system will serve as a starting point of our analysis. Observe first that the system is not overdetermined. Indeed, when both equations are multiplied by a covector q_i, they turn to trivial identities, provided that the electric current is conserved. Thus, we have two linear relations between eight linear equations (3.8) for six independent variables. It means that the rank of the system (3.8) is less than or equal to 6. The physical meaning of this system requires the rank to be exactly equal to 6. Indeed, the unknowns f_{kl} of this system are physically measurable quantities. Thus, for an arbitrary conserved current j^i, they have to be determined from (3.8) uniquely. This physical requirement puts a strong algebraic constraint on the system (3.8) —*its coefficients must form a matrix of a rank of 6*. In fact, it is a constraint on the components of the constitutive pseudo-tensor χ^{ijkl}, whose formal expression we will derive subsequently.

4.2. The homogeneous equation

The homogeneous equation (3.8) is exactly the same as in the standard Maxwell theory. We give here a precise treatment of this equation mostly in order to establish the notation and to illustrate the method used in the following.

Proposition 1. *A most general solution of a linear system*

$$\epsilon^{ijkl} q_j f_{kl} = 0 \tag{4.1}$$

is expressed as

$$f_{kl} = \frac{1}{2}(a_k q_l - a_l q_k), \tag{4.2}$$

where a_k is an arbitrary covector.

Proof. Expression (4.2) is evidently a solution of (4.1). In order to prove that it is a most general solution, we first note that (4.1) is a linear system of four equations for six independent variables f_{ij}. The 4×6 matrix of this system $a^{ikl} = \epsilon^{ijkl} q_j$ with $ik = 01, 02, 03, 12, 23, 31$ and $l = 0, 1, 2, 3$ is given by

$$a^{ikl} = \begin{pmatrix} 0 & 0 & 0 & q_3 & -q_2 & q_1 \\ 0 & -q_3 & q_2 & 0 & 0 & -q_0 \\ q_3 & 0 & -q_1 & 0 & q_0 & 0 \\ q_2 & q_1 & 0 & -q_0 & 0 & 0 \end{pmatrix}. \tag{4.3}$$

The rows of this matrix satisfy a linear relation

$$a^{ikl} q_i = 0. \tag{4.4}$$

Thus its rank is 3 or less. If an arbitrary row is now removed from (4.3), the remaining three columns are assembled in the echelon form. Thus the matrix (4.3) has exactly a rank of 3. Consequently, a general solution of (4.1) has to involve $6 - 3 = 3$ independent parameters. This is exactly what is given in (4.2). Indeed, although the arbitrary covector a_i has four independent components, only three of them are involved in (4.2). In particular, the vector a_i proportional to q_i does not give a contribution. Thus, (4.2) is a most general covariant solution of (4.1). □

It is well known that the homogeneous field equation (3.2) is solved in terms of the standard vector potential A_i. The covector a_i appeared in (4.2) is similar to the Fourier transform of the potential A_i. As usual, this covector is arbitrary if only the homogeneous field equation is taken into account.

4.3. The inhomogeneous equation

Let us now turn to the inhomogeneous equation of (3.8). Substituting the solution (4.2), we arrive at an algebraic system

$$\chi^{ijkl} q_l q_j a_k = j^i. \tag{4.5}$$

Observe that this is a system of four equations for four variables a_k. The matrix of this system,

$$M^{ik} = \chi^{ijkl} q_l q_j, \tag{4.6}$$

will be refereed to as a *characteristic matrix*. We will see that the wave propagation depends exactly on the specific combination of the components of the constitutive pseudo-tensor χ^{ijkl} which are involved in M^{ik}.

When the irreducible decomposition (2.24) is substituted into the characteristic matrix M^{ij}, the completely antisymmetric axion part ${}^{(3)}\chi^{ijkl}$ evidently does not contribute. As for the other two pieces, the principal part is involved only in the symmetric part of the matrix M^{ij}, while the skewon part is involved only in its antisymmetric part. Formally, we can write

$$M^{(ik)} = M^{ik}({}^{(1)}\chi), \qquad M^{[ik]} = M^{ik}({}^{(2)}\chi). \tag{4.7}$$

So, the characteristic matrix M^{ik} is irreducibly decomposed as

$$M^{ik} = M^{(ik)}({}^{(1)}\chi) + M^{[ik]}({}^{(2)}\chi). \tag{4.8}$$

In the characteristic matrix notation, equation (4.5) takes the form

$$M^{ik} a_k = j^i. \tag{4.9}$$

The following two facts will play an important role in our analysis.

(1) *Gauge invariant condition.* Due to the antisymmetry of the constitutive pseudo-tensor χ^{ijkl} in its last two indices, an identity

$$M^{ik} q_k = 0 \tag{4.10}$$

holds true. It is a linear relation between the rows of the matrix M^{ik}. It means that every solution of (4.5) is defined only up to an addition of a term $a_i \sim q_i$. This addition is evidently unphysical since it does not contribute to the electromagnetic strength. Consequently, relation (4.10) has to be interpreted as a gauge invariant condition.

(2) *Charge conservation condition.* Another evident identity for the matrix M^{ik} emerges from the antisymmetry of the constitutive pseudo-tensor χ^{ijkl} in its first two indices:

$$M^{ik} q_i = 0. \tag{4.11}$$

It is a linear relation between the columns of the matrix M^{ik}. Being compared with (4.9), it yields $j^i q_i = 0$. Consequently, relation (4.11) has to be interpreted as a charge conservation condition.

Thus we arrive at some type of a duality between the charge conservation and the gauge invariance. Note that this duality is expressed by a standard algebraic fact: for any matrix, the column rank and the row rank are equal to one another.

Due to the conditions indicated above, the rows (and the columns) of the matrix M^{ij} are linearly dependent, so its determinant is equal to zero. It can be checked straightforwardly, but one has to apply here rather tedious calculations.

5. Dispersion relation

5.1. How it emerges

In the vacuum case of the free electromagnetic waves, (4.5) takes a form of a linear homogeneous system of four equations for four components of the covector a_k:

$$\chi^{ijkl}q_lq_ja_k = 0, \qquad \text{or} \qquad M^{ik}a_k = 0. \tag{5.1}$$

The gauge relation (4.10) can be interpreted as a fact that

$$a_l = Cq_l \tag{5.2}$$

is a formal solution of (5.1). This solution does not contribute to the electromagnetic field strength so it is unphysical. Hence, the formal system (5.1) can have a nonzero solution, only if it has an *additional solution* which must be linearly independent on (5.2). Consequently (5.1) must have at least two linearly independent solutions. A known fact from linear algebra is that a linear system has two (or more) linearly independent solutions if and only if the rank of the matrix M^{ij} is 2 (or less). In this case, the adjoint matrix (constructed from the cofactors of M^{ij}) is equal to zero, $A_{ij} = 0$.

In order to present a formal expression of this fact we will start with a formula for the determinant of an arbitrary fourth-order matrix:

$$\det(M) = \frac{1}{4!}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}M^{ij}M^{i_1j_1}M^{i_2j_2}M^{i_3j_3}. \tag{5.3}$$

The components of the adjoint matrix are expressed by the derivatives of the determinant relative to the entries of the matrix:

$$A_{ij} = \frac{\partial \det(M)}{\partial M^{ij}}. \tag{5.4}$$

Explicitly,

$$A_{ij} = \frac{1}{3!}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}M^{i_1j_1}M^{i_2j_2}M^{i_3j_3}. \tag{5.5}$$

Consequently, we derived a physically motivated condition on the components of the constitutive pseudo-tensor χ_{ijkl}.

Theorem 2. *The Maxwell system with a general linear constitutive relation has a non-trivial wave-type solution if and only if the adjoint of the matrix $M^{ik} = \chi^{ijkl}q_lq_j$ is equal to zero, i.e.*

$$A_{ij} = 0, \tag{5.6}$$

or explicitly,

$$\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}M^{i_1j_1}M^{i_2j_2}M^{i_3j_3} = 0. \tag{5.7}$$

Since the adjoint matrix has 16 independent components, it seems that we have to require, in general, 16 independent conditions. In fact, the situation is rather simpler. The following algebraic fact is important in our analysis, so we present here its formal proof.

Proposition 3. *Let a square $n \times n$ matrix M^{ij} satisfies the relations*

$$M^{ij}q_i = 0, \qquad M^{ij}q_j = 0, \tag{5.8}$$

for an arbitrary nonzero n-covector q_i. The adjoint matrix $A_{ij} = Adj(M^{ij})$ is proportional to the tensor square of q_i, i.e.

$$A_{ij} = \lambda(q)q_iq_j. \tag{5.9}$$

Proof. Due to (5.8), the rows (and the columns) of M^{ij} are linear dependent, so the matrix is singular and its rank is equal to $n-1$ or less. If the rank is less than $(n-1)$, the adjoint matrix is identically zero and (5.9) is satisfied trivially for $\lambda = 0$.

Let the rank of M^{ij} be equal to $(n-1)$. In this case, q_i is a *unique covector* (up to a multiplication on a constant) that satisfies (5.8). It is well known that for a matrix of a rank of $(n-1)$, the adjoint A_{ij} is a matrix of a rank of 1. Moreover, an arbitrary rank 1 matrix can be written as a tensor product of two covectors:

$$A_{ij} = u_i v_j. \tag{5.10}$$

Let us now show that both these covectors must be proportional to q_i. Indeed, the product of an arbitrary matrix with its adjoint is equal to the determinant of the matrix times the unit matrix (this is the generalized Laplace expansion theorem). In our case, M^{ij} is a singular matrix so

$$A_{ij}M^{ik} = A_{ij}M^{kj} = 0. \tag{5.11}$$

Substituting here (5.10), we have the relations

$$u_i v_j M^{ik} = 0 \qquad u_i v_j M^{jk} = 0, \tag{5.12}$$

or, equivalently,

$$u_i M^{ik} = 0 \qquad v_j M^{jk} = 0. \tag{5.13}$$

Comparing this pair of relations to (5.9) and remembering that q_i is unique up to a multiplication on a constant, we conclude that both covectors u_i and v_i are proportional to q_i. Thus (5.4) indeed takes the form (5.9). □

Consequently, in order to have a physically non-trivial vacuum wave-type solution, the system (5.1) has to satisfy a unique scalar condition

$$\lambda(q) = 0. \tag{5.14}$$

In fact, this is an expression for the principal *dispersion relation*.

5.2. Some basic properties of the dispersion relation

Even without an explicit expression for the function $\lambda(q)$, we are now ready to derive some characteristic properties of the dispersion relation (5.14). Some of these properties were recently derived in [19] by involved straightforward calculations. In our approach, these properties are immediate consequences of the definition of the λ-function (5.9).

Corollary 4. *$\lambda(q)$ is a homogeneous fourth-order polynomial of the wave covector q_i, i.e.*

$$\lambda(q) = \mathcal{G}^{ijkl} q_i q_j q_k q_l, \tag{5.15}$$

where $\mathcal{G}^{ijkl}$ is a pseudo-tensor independent of q_i.

Indeed, the adjoint matrix is a homogeneous polynomial of the sixth order in q_i. By (5.9), after extracting the product $q_i q_j$ we remain with a sum of terms each of which is a product of four components of the covector q_i. Since $\lambda(q)$ is a pseudo-scalar, $\mathcal{G}^{ijkl}$ is a pseudo-tensor.

Corollary 5. *$\lambda(q)$ is a homogeneous third-order polynomial of the constitutive pseudo-tensor χ^{ijkl}.*

Indeed, the adjoint matrix is a sum of terms which are cubic in the matrix M^{ij}. Every such term is a product of three χ's which remains also after extracting the product $q_i q_j$ on the right-hand side of (5.9).

Corollary 6. *The equation $\lambda(q) = 0$ defines a complex algebraic cone.*

For a given set of vectors $K = \{x_1, x_2, \ldots\}$, an algebraic cone is defined as a set of vectors $CK = \{Cx_1, Cx_2, \ldots\}$, where $C > 0$ is an arbitrary number. Due to the homogeneity of the dispersion relation (5.14), each of its solutions is defined up to a product on a constant. This (generally complex) algebraic cone is a prototype of a light cone emerging from a Lorentz metric of vacuum electrodynamics or by optical metric of electromagnetism in dielectric media. The algebraic cone is real when additional hyperbolicity conditions are applied [27].

Corollary 7. *The axion part of the constitutive tensor does not contribute to the function $\lambda(q)$. In other words,*

$$\lambda({}^{(1)}\chi + {}^{(2)}\chi + {}^{(3)}\chi) = \lambda({}^{(1)}\chi + {}^{(2)}\chi). \tag{5.16}$$

Indeed, the axion part does not contribute to the matrix M^{ij}, so it does not appear in its adjoint.

Corollary 8. *The skewon part alone does not emerge in a non-trivial dispersion relation. In other words, a relation*

$$\lambda({}^{(2)}\chi) = 0 \tag{5.17}$$

holds identically.

Indeed, in order to have a non-trivial (non-zero) expression for $\lambda(q)$, the rank of the matrix M^{ij} has to be equal to 3. The skewon part generates an antisymmetric matrix $M^{[ij]}$. Since the rank of an arbitrary antisymmetric matrix is even, the skewon part alone does not emerge in a non-trivial dispersion relation.

Corollary 9. *A non-trivial (non-zero) dispersion relation emerges only if the principal part of the constitutive tensor is non-zero:*

$${}^{(1)}\chi \neq 0. \tag{5.18}$$

This is an immediate result of the previous statements.

6. Dispersion relation in an explicit form

6.1. Covariant dispersion relation I

Our task now is to derive an explicit expression for the dispersion relation. Recall that it is represented by a scalar equation

$$\lambda(q) = 0, \tag{6.1}$$

where the function $\lambda(q)$ satisfies the equation

$$A_{ij} = \lambda(q) q_i q_j \tag{6.2}$$

for the adjoint matrix A^{ij} of the characteristic matrix M^{ij}. To have an explicit expression for $\lambda(q)$, it is necessary 'to divide' both sides of (6.2) by the product $q_i q_j$. Certainly such a 'division' must be produced in a covariant way. We will look first for a solution of this problem in a special coordinate system. Let a zeroth (time) axis be directed as a wave covector, i.e. $q_0 = q, q_1 = q_2 = q_3 = 0$. Substituting into (6.2) we have

$$\lambda(q) q^2 = \frac{1}{3!} \epsilon_{0 i_1 i_2 i_3} \epsilon_{0 j_1 j_2 j_3} M^{i_1 j_1} M^{i_2 j_2} M^{i_3 j_3}. \tag{6.3}$$

Due to the symmetry properties of the Lévi-Civitá pseudo-tensor, all four-dimensional indices can be replaced by the three-dimensional ones ($\alpha, \beta = 1, 2, 3$). So we get

$$\lambda(q)q^2 = \frac{1}{3!}\epsilon_{0\alpha_1\alpha_2\alpha_3}\epsilon_{0\beta_1\beta_2\beta_3}M^{\alpha_1\beta_1}M^{\alpha_2\beta_2}M^{\alpha_3\beta_3}. \tag{6.4}$$

In the chosen system, the non-zero components of the matrix M^{ij} are

$$M^{\alpha\beta} = \chi^{\alpha m\beta n}q_m q_n = \chi^{\alpha 0\beta 0}q^2. \tag{6.5}$$

Consequently,

$$\lambda(q) = \frac{1}{3!}\epsilon_{\alpha_1\alpha_2\alpha_3}\epsilon_{\beta_1\beta_2\beta_3}\chi^{\alpha_1 0\beta_1 0}\chi^{\alpha_2 0\beta_2 0}\chi^{\alpha_3 0\beta_3 0}q^4, \tag{6.6}$$

where the three-dimensional Lévi-Civitá pseudo-tensor $\epsilon_{\alpha_1\alpha_2\alpha_3} = \epsilon_{0\alpha_1\alpha_2\alpha_3}$ is involved. The non-covariant dispersion relation takes the form [19]

$$\epsilon_{\alpha_1\alpha_2\alpha_3}\epsilon_{\beta_1\beta_2\beta_3}\chi^{\alpha_1 0\beta_1 0}\chi^{\alpha_2 0\beta_2 0}\chi^{\alpha_3 0\beta_3 0}q^4 = 0. \tag{6.7}$$

In [15], [19], equation (6.7) was derived by the consideration of the three-dimensional determinant of the system. It was generalized to a covariant four-dimensional dispersion relation

$$\frac{1}{4!}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}\chi^{ii_1ja}\chi^{bi_2j_1c}\chi^{di_3j_2j_3}q_aq_bq_cq_d = 0. \tag{6.8}$$

The λ-function can be read off from this equation as

$$\lambda(q) = \frac{1}{4!}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}\chi^{ii_1ja}\chi^{bi_2j_1c}\chi^{di_3j_2j_3}q_aq_bq_cq_d. \tag{6.9}$$

The result (6.8) turns to be correct, as we will show in the following.

6.2. Covariant dispersion relation II

We will now give a pure covariant derivation of the scalar function $\lambda(q)$ and of the corresponding dispersion relation. Differentiation of (5.9) relative to the components of the covector q_m yields

$$\frac{\partial A_{ij}}{\partial q_m} = \frac{\partial\lambda(q)}{\partial q_m}q_iq_j + \lambda(q)\big(\delta_i^m q_j + \delta_j^m q_i\big). \tag{6.10}$$

Let us contract this equation over the indices m and i and use the Euler formula for a fourth-order homogeneous function $\lambda(q)$. Consequently, we derive

$$\frac{\partial A_{ij}}{\partial q_i} = 9\lambda(q)q_j. \tag{6.11}$$

A second-order derivative of this expression is given by

$$\frac{\partial^2 A_{ij}}{\partial q_i\partial q_m} = 9\left(\frac{\partial\,\lambda(q)}{\partial q_m}q_j + \lambda(q)\delta_j^m\right). \tag{6.12}$$

Now summing over the indices m and j and using once more the Euler formula, we derive

$$\lambda(q) = \frac{1}{72}\,\frac{\partial^2 A_{ij}}{\partial q_i\partial q_j}. \tag{6.13}$$

Consequently, we have proved the following.

Theorem 10. *For the Maxwell system with a general local linear constitutive relation, the dispersion relation is given by*

$$\frac{\partial^2 A_{ij}}{\partial q_i\partial q_j} = 0. \tag{6.14}$$

In order to have an expression of the λ-function in terms of the matrix M^{ij}, we calculate the derivatives of the adjoint matrix

$$\frac{\partial A_{ij}}{\partial q_i} = \frac{1}{2}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}\frac{\partial M^{i_1j_1}}{\partial q_i}M^{i_2j_2}M^{i_3j_3}. \tag{6.15}$$

Substituting into (6.14) and calculating the second-order derivative, we get

$$\lambda(q) = \frac{1}{144}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}\left(\frac{\partial^2 M^{i_1j_1}}{\partial q_i\partial q_j}M^{i_2j_2} + 2\frac{\partial M^{i_1j_1}}{\partial q_i}\frac{\partial M^{i_2j_2}}{\partial q_j}\right)M^{i_3j_3}. \tag{6.16}$$

This expression may be useful for actual calculations of the dispersion relation for different electromagnetic media. In particular, the following decomposition represents the contribution of the skewon part in the dispersion relation. Different forms of it can be found in [19].

Proposition 12. *Due to the irreducible decomposition of the constitutive pseudo-tensor, the dispersion relation* $\lambda(\chi) = 0$ *is given by*

$$\lambda({}^{(1)}\chi) + \frac{1}{2}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}\frac{\partial^2}{\partial q_i\partial q_j}[M^{(i_1j_1)}({}^{(1)}\chi)M^{[i_2j_2]}({}^{(2)}\chi)M^{[i_3j_3]}({}^{(2)}\chi)] = 0. \tag{6.17}$$

Proof. The relation follows straightforwardly when the decomposition (4.8) is substituted into (6.14) and the antisymmetric terms are removed. □

6.3. *Covariant dispersion relation III*

An explicit expression of the λ-function via the constitutive pseudo-tensor is calculated by the derivatives of the matrix

$$M^{ij} = \chi^{iajb}q_aq_b = -\chi^{iabj}q_aq_b = -\chi^{i(ab)j}q_aq_b. \tag{6.18}$$

The first-order derivative is given by

$$\frac{\partial M^{i_1j_1}}{\partial q_i} = \frac{\partial}{\partial q_i}(\chi^{i_1mj_1n}q_mq_n) = -2\chi^{i_1(im)j_1}q_m. \tag{6.19}$$

Hence, the second-order derivative reads

$$\frac{\partial^2 M^{i_1j_1}}{\partial q_i\partial q_j} = -2\chi^{i_1(ij)j_1}. \tag{6.20}$$

Consequently, the left-hand side of (6.16) takes the form

$$\begin{aligned}\lambda(q) &= \frac{1}{3\cdot 4!}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}(-\chi^{i_1(ij)j_1}M^{i_2j_2} + 4\chi^{i_1(ia)j_1}\chi^{i_2(jb)j_2}q_aq_b)M^{i_3j_3}\\ &= \frac{1}{3\cdot 4!}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}(\chi^{i_1(ij)j_1}\chi^{i_2(ab)j_2} + 4\chi^{i_1(ia)j_1}\chi^{i_2(jb)j_2})M^{i_3j_3}q_aq_b\\ &= \frac{1}{3\cdot 4!}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}(\chi^{i_1(ij)j_1}\chi^{i_2(ab)j_2} + 4\chi^{i_1(ia)j_1}\chi^{i_2(jb)j_2})\chi^{i_3(cd)j_3}q_aq_bq_cq_d.\end{aligned} \tag{6.21}$$

We finally have the covariant dispersion relation in the form

$$\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}(\chi^{i_1(ij)j_1}\chi^{i_2abj_2} + 4\chi^{i_1(ia)j_1}\chi^{i_2(jb)j_2})\chi^{i_3cdj_3}q_aq_bq_cq_d = 0. \tag{6.22}$$

This equation turns out to be equivalent to (6.8). The direct proof of this fact was provided by Yu Obukhov, see the appendix.

Different forms of the covariant dispersion relation are additional outputs of this proof. In particular, by using the Y_6 term (A.7) we have the dispersion relation in the form

$$\epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1aij_1}\chi^{i_2bjj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d = 0. \tag{6.23}$$

The Y_3 term (A.4) gives

$$\epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1iaj_1}\,\chi^{i_2jbj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d = 0. \tag{6.24}$$

Probably the most symmetric form is obtained from the Y_1 term (A.2):

$$\epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{ii_1jj_1}\,\chi^{i_2abj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d = 0. \tag{6.25}$$

7. Maxwell electrodynamics reinstated

7.1. Maxwell constitutive pseudo-tensor

When the premetric scheme is applied on a manifold with a prescribed metric tensor g_{ij}, the standard Maxwell electrodynamics (2.5) is reinstated by substitutions

$$\mathcal{H}^{ij} = \sqrt{-g}\,g^{im}g^{jn}F_{mn} = \tfrac{1}{2}\sqrt{-g}(g^{im}g^{jn} - g^{in}g^{jm})F_{mn}. \tag{7.1}$$

Recall that the electric current vector density is expressed as

$$\mathcal{J}^i = J^i\sqrt{-g}. \tag{7.2}$$

The constitutive relation (7.1) corresponds to a choice of a special *Maxwell–Lorentz constitutive pseudo-tensor*

$$^{(\mathtt{Max})}\chi^{ijkl} = \sqrt{-g}(g^{ik}g^{jl} - g^{il}g^{jk}). \tag{7.3}$$

We use here a system of units in which a constant with the dimension of an admittance (denoted by λ_0 in [19]) is taken to be equal to 1. When (7.3) is substituted in (2.21) we return to the standard Maxwell electrodynamics system (2.5).

7.2. Dispersion relation

Let us derive the standard metrical dispersion relation. The characteristic matrix corresponded to the constitutive pseudo-tensor (7.3) takes the form

$$M^{ik} = \sqrt{|g|}(g^{ik}q^2 - q^iq^j), \tag{7.4}$$

where the notations $q^2 = g^{ij}q_iq_j$ and $q^i = g^{im}q_m$ are used. The adjoint of this matrix is calculated straightforwardly:

$$\begin{aligned} A_{ij} &= \frac{1}{3!}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}M^{i_1j_1}M^{i_2j_2}M^{i_3j_3} \\ &= \frac{1}{3!}\sqrt{|g|^3}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}(g^{i_1j_1}q^2 - q^{i_1}q^{j_1})(g^{i_2j_2}q^2 - q^{i_2}q^{j_2})(g^{i_3j_3}q^2 - q^{i_3}q^{j_3}) \\ &= \frac{1}{3!}\sqrt{|g|^3}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}(g^{i_1j_1}g^{i_2j_2}g^{i_3j_3}q^6 - 3g^{i_2j_2}g^{i_3j_3}q^{i_1}q^{j_1}q^4) \\ &= \frac{1}{3!}\sqrt{|g|^3}[-6g_{ij}q^6 + 6(g_{ij}g_{i_1j_1} - g_{ij_1}g_{i_1j})q^{i_1}q^{j_1}q^4]. \end{aligned} \tag{7.5}$$

The first two terms cancel one another so the adjoint matrix remains in the form

$$A_{ij} = -\sqrt{|g|^3}q_iq_jq^4. \tag{7.6}$$

Correspondingly,

$$\lambda = -\sqrt{|g|^3}q^4 \tag{7.7}$$

and the dispersion relation takes the standard form

$$q^2 = 0 \quad\Longleftrightarrow\quad g^{ij}q_iq_j = 0. \tag{7.8}$$

From (7.7) we also deduce that the pseudo-tensor appeared in (5.15) takes the form

$$\mathcal{G}^{ijkl} = -\tfrac{1}{2}\sqrt{|g|^3}(g^{ij}g^{kl} + g^{ik}g^{jl}). \tag{7.9}$$

8. Results and discussion

Premetric electrodynamics can be viewed as a general framework for the description of a wide class of electromagnetic effects. In this paper we discussed the geometric optics approximation of the wave propagation in this model. Due to the standard procedure of partial differential equations theory, such approximation represents the leading contribution to the corresponding solutions. We derived a covariant dispersion relation and showed that this relation represents the existence of the wave-type solution in the premetric electromagnetic model. It should be noted that our expression of the covariant dispersion relation is not less complicated than the one represented in the literature [19]. An advantage of our approach is that we give a straightforward covariant procedure how the dispersion relation can be derived for various constitutive relations. For this, one does not need to deal with the explicit covariant formula at all. It is enough to construct the characteristic matrix M^{ij} and calculate its adjoint A_{ij}. Due to the proposition, proved in the paper, the extra factors q_iq_j are separated from A_{ij} and the remain part is the essential term of the dispersion relation. We have shown how this procedure works in the case of the standard Maxwell constitutive relation. The problem of uniqueness of the wave-type solution in the premetric electromagnetic model is related to another principal notion—the photon propagator [8, 28]. A detailed consideration of this quantity and of its relation to the uniqueness problem will be represented in a separate publication.

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Appendix. Obukhov's proof of the equivalence of the dispersion relations

A.1. Expressions for comparison

Equation (6.21) reads

$$\lambda(q) = \frac{1}{6\cdot 4!}[Y_1 + Y_2 + 2(Y_3 + Y_4 + Y_5 + Y_6)], \tag{A.1}$$

where the six terms are explicitly given by

$$Y_1 := \epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1ijj_1}\,\chi^{i_2abj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d, \tag{A.2}$$

$$Y_2 := \epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1jij_1}\,\chi^{i_2abj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d, \tag{A.3}$$

$$Y_3 := \epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1iaj_1}\,\chi^{i_2jbj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d, \tag{A.4}$$

$$Y_4 := \epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1iaj_1}\,\chi^{i_2bjj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d, \tag{A.5}$$

$$Y_5 := \epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1aij_1}\,\chi^{i_2jbj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d, \tag{A.6}$$

$$Y_6 := \epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1aij_1}\,\chi^{i_2bjj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d. \tag{A.7}$$

Our Fresnel equation is given by formulas (D.2.22) and (D.2.23) of the book [19], with

$$\begin{aligned}\widetilde{\lambda}(q) &= = \frac{1}{4!}\hat{\epsilon}_{mnpq}\,\hat{\epsilon}_{rstu}\,\widetilde{\chi}^{mnri}\,\widetilde{\chi}^{jpsk}\,\widetilde{\chi}^{lqtu}\,q_iq_jq_kq_l\\ &= \frac{1}{4!}\hat{\epsilon}_{ii_1i_3i_2}\,\hat{\epsilon}_{j_1j_3jj_2}\,\widetilde{\chi}^{ii_1j_1a}\,\widetilde{\chi}^{bi_3j_3c}\,\widetilde{\chi}^{di_2jj_2}\,q_aq_bq_cq_d\\ &= -\frac{1}{4!}\,\hat{\epsilon}_{ii_1i_2i_3}\,\hat{\epsilon}_{jj_1j_2j_3}\,\widetilde{\chi}^{i_iiaj_1}\,\widetilde{\chi}^{i_3cdj_3}\,\widetilde{\chi}^{i_2bjj_2}\,q_aq_bq_cq_d.\end{aligned} \tag{A.8}$$

Formulas (A.2)–(A.7) also contain the reduced constitutive tensor $\widetilde{\chi} = {}^{(1)}\chi + {}^{(2)}\chi$ since the axion part ${}^{(3)}\chi$ does not contribute to the matrix M^{ij}. So from now on, we will drop the tildes, simply keeping in mind that the axion is not present in all our derivations.

Accordingly, we find that

$$\tilde{\lambda}(q) = -\frac{1}{4!}Y_4. \tag{A.9}$$

A.2. Method

We will use the well-known fact that in four dimensions any totally antisymmetric tensor of the fifth rank is identically zero. In particular,

$$\epsilon_{[ijkl}\,A_{a]} \equiv 0, \tag{A.10}$$

where A_a can be anything (the other indices are suppressed for clarity).

Since the Lévi-Civitá tensor is totally antisymmetric in its four indices, the above identity contains just five terms, and we can conveniently rewrite it as follows:

$$\epsilon_{ijkl}\,A_a \equiv \epsilon_{ajkl}\,A_i + \epsilon_{iakl}\,A_j + \epsilon_{ijal}\,A_k + \epsilon_{ijka}\,A_l. \tag{A.11}$$

We will repeatedly use this identity in order to establish the relations between different terms Y_1–Y_6.

A.3. Relations between different terms

Derivation of relations between Y_1–Y_6 is technically simple, but requires patience and attention. The main tool is the identity (A.11). In particular, we have

$$\epsilon_{ii_1i_2i_3}\,q_a \equiv \epsilon_{ai_1i_2i_3}\,q_i + \epsilon_{iai_2i_3}\,q_{i_1} + \epsilon_{ii_1ai_3}\,q_{i_2} + \epsilon_{ii_1i_2a}\,q_{i_3}, \tag{A.12}$$

$$\epsilon_{jj_1j_2j_3}\,q_a \equiv \epsilon_{aj_1j_2j_3}\,q_j + \epsilon_{jaj_2j_3}\,q_{j_1} + \epsilon_{jj_1aj_3}\,q_{j_2} + \epsilon_{jj_1j_2a}\,q_{j_3}, \tag{A.13}$$

$$\epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3} \equiv \epsilon_{ji_1i_2i_3}\,\epsilon_{ij_1j_2j_3} + \epsilon_{iji_2i_3}\,\epsilon_{i_1j_1j_2j_3} + \epsilon_{ii_1ji_3}\,\epsilon_{i_2j_1j_2j_3} + \epsilon_{ii_1i_2j}\,\epsilon_{i_3j_1j_2j_3}. \tag{A.14}$$

These three formulas are all we need in the subsequent computations.

Relation between Y_6 and Y_4. Using (A.12) in (A.7), we find

$$\begin{aligned}
Y_6 &= \epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1aij_1}\,\chi^{i_2bjj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d\\
&= \epsilon_{ai_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1aij_1}\,\chi^{i_2bjj_2}\,\chi^{i_3cdj_3}\,q_iq_bq_cq_d\\
&\quad + \epsilon_{iai_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1aij_1}\,\chi^{i_2bjj_2}\,\chi^{i_3cdj_3}\,q_{i_1}q_bq_cq_d\\
&\quad + \epsilon_{ii_1ai_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1aij_1}\,\chi^{i_2bjj_2}\,\chi^{i_3cdj_3}\,q_{i_2}q_bq_cq_d\\
&\quad + \epsilon_{ii_1i_2a}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1aij_1}\,\chi^{i_2bjj_2}\,\chi^{i_3cdj_3}\,q_{i_3}q_bq_cq_d.
\end{aligned} \tag{A.15}$$

The last two terms are zero because the symmetric tensors $q_{i_2}q_b$ and $q_{i_3}q_c$ are contracted with skew-symmetric pairs of indices. The first term, after renaming the summation indices $a \to i$ and $i \to a$, is equal to Y_4. The second term, after renaming the summation indices $a \to i_1$ and $i_1 \to a$, is equal to the original expression with the different sign, i.e. to $-Y_6$. Consequently,

$$Y_6 = \tfrac{1}{2}\,Y_4. \tag{A.16}$$

Relation between Y_3 and Y_5. Analogously, using (A.12) in (A.4), we find

$$\begin{aligned} Y_3 &= \epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1iaj_1}\chi^{i_2jbj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d \\ &= \epsilon_{ai_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1iaj_1}\chi^{i_2jbj_2}\,\chi^{i_3cdj_3}\,q_iq_bq_cq_d \\ &\quad + \epsilon_{iai_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1iaj_1}\chi^{i_2jbj_2}\,\chi^{i_3cdj_3}\,q_{i_1}q_bq_cq_d \\ &\quad + \epsilon_{ii_1ai_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1iaj_1}\chi^{i_2jbj_2}\,\chi^{i_3cdj_3}\,q_{i_2}q_bq_cq_d \\ &\quad + \epsilon_{ii_1i_2a}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1iaj_1}\chi^{i_2jbj_2}\,\chi^{i_3cdj_3}\,q_{i_3}q_bq_cq_d. \end{aligned} \tag{A.17}$$

The last term vanishes because $q_{i_3}q_c$ is contracted with the skew-symmetric pair of indices. The first term, after renaming the summation indices $a \to i$ and $i \to a$, is equal to Y_5. The second term, after renaming the summation indices $a \to i$, $i \to i_1$ and $i_1 \to a$, is again equal to Y_5. Finally, the third term will be denoted by Δ. Consequently, we find

$$Y_3 = 2\,Y_5 + \Delta. \tag{A.18}$$

Relation between Y_3 and Y_1. Now, if we use (A.13) in (A.4), we find

$$\begin{aligned} Y_3 &= \epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1iaj_1}\chi^{i_2jbj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d \\ &= \epsilon_{ii_1i_2i_3}\,\epsilon_{aj_1j_2j_3}\,\chi^{i_1iaj_1}\chi^{i_2jbj_2}\,\chi^{i_3cdj_3}\,q_jq_bq_cq_d \\ &\quad + \epsilon_{ii_1i_2i_3}\,\epsilon_{jaj_2j_3}\,\chi^{i_1iaj_1}\chi^{i_2jbj_2}\,\chi^{i_3cdj_3}\,q_{j_1}q_bq_cq_d \\ &\quad + \epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1aj_3}\,\chi^{i_1iaj_1}\chi^{i_2jbj_2}\,\chi^{i_3cdj_3}\,q_{j_2}q_bq_cq_d \\ &\quad + \epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2a}\,\chi^{i_1iaj_1}\chi^{i_2jbj_2}\,\chi^{i_3cdj_3}\,q_{j_3}q_bq_cq_d. \end{aligned} \tag{A.19}$$

The last two terms are zero because the symmetric tensors $q_{j_2}q_b$ and $q_{j_3}q_d$ are contracted with skew-symmetric pairs of indices. The first term, after renaming the summation indices $a \to j$ and $j \to a$, is equal to Y_1. The second term, after renaming the summation indices $a \to j_1$ and $j_1 \to a$, is equal to the original expression with the different sign, i.e. to $-Y_3$. Consequently,

$$Y_3 = \tfrac{1}{2}\,Y_1. \tag{A.20}$$

Relation between Y_4 and Y_1. Now, if we use (A.13) in (A.5), we find

$$\begin{aligned} Y_4 &= \epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1iaj_1}\chi^{i_2bjj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d \\ &= \epsilon_{ii_1i_2i_3}\,\epsilon_{aj_1j_2j_3}\,\chi^{i_1iaj_1}\chi^{i_2bjj_2}\,\chi^{i_3cdj_3}\,q_jq_bq_cq_d \\ &\quad + \epsilon_{ii_1i_2i_3}\,\epsilon_{jaj_2j_3}\,\chi^{i_1iaj_1}\chi^{i_2bjj_2}\,\chi^{i_3cdj_3}\,q_{j_1}q_bq_cq_d \\ &\quad + \epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1aj_3}\,\chi^{i_1iaj_1}\chi^{i_2bjj_2}\,\chi^{i_3cdj_3}\,q_{j_2}q_bq_cq_d \\ &\quad + \epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2a}\,\chi^{i_1iaj_1}\chi^{i_2bjj_2}\,\chi^{i_3cdj_3}\,q_{j_3}q_bq_cq_d. \end{aligned} \tag{A.21}$$

The last term vanishes because $q_{j_3}q_d$ is contracted with the skew-symmetric pair of indices. The first term, after renaming the summation indices $a \to j$ and $j \to a$, is equal to Y_1. The second term, after renaming the summation indices $a \to j_1$ and $j_1 \to a$, is equal to the original term with a minus sign, i.e. to $-Y_4$. Finally, the third term, after renaming the summation indices $a \to j$, $j \to j_2$ and $j_2 \to a$ is again equal to Y_1. Consequently,

$$Y_4 = Y_1. \tag{A.22}$$

Relation between Y_2 and Y_1. Finally, we need one more relation. This is obtained when we use (A.14) in (A.3). Then

$$\begin{aligned} Y_2 &= \epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1jij_1}\,\chi^{i_2abj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d \\ &= \epsilon_{ji_1i_2i_3}\,\epsilon_{ij_1j_2j_3}\,\chi^{i_1jij_1}\,\chi^{i_2abj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d \\ &\quad + \epsilon_{iji_2i_3}\,\epsilon_{i_1j_1j_2j_3}\,\chi^{i_1jij_1}\,\chi^{i_2abj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d \\ &\quad + \epsilon_{ii_1ji_3}\,\epsilon_{i_2j_1j_2j_3}\,\chi^{i_1jij_1}\,\chi^{i_2abj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d \\ &\quad + \epsilon_{ii_1i_2j}\,\epsilon_{i_3j_1j_2j_3}\,\chi^{i_1jij_1}\,\chi^{i_2abj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d. \end{aligned} \tag{A.23}$$

Here, all terms are nonvanishing. The first term, after renaming the summation indices $i \to j$ and $j \to i$, is equal to Y_1. The second term, after renaming the summation indices $i_1 \to j$ and $j \to i_1$, is equal to the original term with a minus sign, i.e. to $-Y_2$. And the two last terms are both equal to Δ. Accordingly, we find

$$Y_2 = \tfrac{1}{2}\,Y_1 + \Delta. \tag{A.24}$$

Appendix A.4. Final result

Now we can collect all the intermediate relations (A.16), (A.18), (A.20), (A.22) and (A.24) into the following list:

$$Y_1 = Y_4, \tag{A.25}$$

$$Y_2 = \tfrac{1}{2}Y_4 + \Delta, \tag{A.26}$$

$$Y_3 = \tfrac{1}{2}Y_4, \tag{A.27}$$

$$Y_4 = Y_4, \tag{A.28}$$

$$Y_5 = \tfrac{1}{4}Y_4 - \tfrac{1}{2}\,\Delta, \tag{A.29}$$

$$Y_6 = \tfrac{1}{2}Y_4. \tag{A.30}$$

From these we now derive the final result for the dispersion relation:

$$Y_1 + Y_2 + 2(Y_3 + Y_4 + Y_5 + Y_6) = 6Y_4. \tag{A.31}$$

Thus,

$$\lambda = \frac{1}{6\cdot 4!}[Y_1 + Y_2 + 2(Y_3 + Y_4 + Y_5 + Y_6)] = \frac{1}{4!}Y_4 = -\tilde{\lambda}. \tag{A.32}$$

Summarizing, the Fresnel equations obtained by different methods *agree completely*.

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